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Common Tangent Space $\mathbb{R}_1^4$ from $U(2)$ Charges

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It is discussed how a common space-time can be constructed from a proposed hidden $U(2)$ world. Schrödinger's idea to obtain discrete eigenvalues by solving the Maxwell equations for the field F on compact spaces without boundaries is modified by orthogonality and identification concepts for the potentials A. Using residue classes with respect to the metric (Clifford algebra), a common spinor space $\mathbb{C}^4 = R \oplus L$ and a common Minkowski tangent space $\mathbb{R}_1^4$ are bilinearly constructed from tangent spaces of $U(2)$ individuals [$U(2)$ manifolds with orthogonal potentials]. The space constructed has the following properties. (1) There are algebraic elements for the identification of $U(2)$ individuals from $\mathbb{R}_1^4$ as spinors ψ and vectors A. (2) The transfer of the potentials from $U(2)$ via $\mathbb{C}^4$ to $\mathbb{R}_1^4$ is linear. (3) The hidden $U(2)$ content of the left- and right-handed spaces (L, R) is quite different. The potentials on $U(2)$ individuals are transformed into complex wave functions ψ on the spinor space and into 1-forms A on $\mathbb{R}_1^4$ that can be enlarged to gauge potentials. The construction is discussed from an old point of view of Einstein's, starting with the electric charge as the primary concept for quantum theory. The construction of the tangent space $\mathbb{R}_1^4$ does not depend on a preceding introduction of any points (uncertainty). The identity problem of the interpretation of the quantum theory is discussed in some detail. It is indicated how the algebraic, partially *ad hoc* constructions can give a rigid frame for further analytical work.

1. INTRODUCTION

This paper is intended to be a further step in the realization of an old idea of Einstein's (1909) of starting with the electric charge as the primary concept for quantum theory.

In previous papers (Donth, 1984, 1986; Donth and Lange, 1986) a hidden nonlocal charge model was constructed by using the periodic solutions of the following conjugate transverse wave equations for potentials A_i, $i = 1, 2$, on $U(2)$:

$$\frac{\partial^2 A_1}{\partial \tau^2} - \frac{1}{\cos^2 \vartheta_3} \frac{\partial^2 A_1}{\partial \varphi_2^2} - \frac{1}{\cot \vartheta_3} \frac{\partial}{\partial \vartheta_3}\left(\cot \vartheta_3 \frac{\partial A_1}{\partial \vartheta_3}\right) = 0 \tag{1a}$$

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$$\frac{\partial^2 A_2}{\partial \tau^2} - \frac{1}{\sin^2 \vartheta_3} \frac{\partial^2 A_2}{\partial \varphi_1^2} - \frac{1}{\tan \vartheta_3} \frac{\partial}{\partial \vartheta_3}\left(\tan \vartheta_3 \frac{\partial A_2}{\partial \vartheta_3}\right) = 0 \tag{1b}$$

The symbols will be explained in the next section. Thus, our approach starts with an idea of Schrödinger's (1940) to obtain discrete eigenvalues by solving Maxwell's equations for the Faraday field F on a compact space without boundaries, e.g., on the Einstein cosmos [$\sim U(2)$]. Two modifying concepts are introduced and will be generally used.

(A) The concept of *orthogonality* (and related concepts such as transversality) are used, e.g., for "narrowing" the Maxwell or harmonic equations for the 1-form A on $U(2)$ to equations (1a), (1b) ($\text{div}_4 A = 0$, $g_{w12}A_1A_2 = 0$), and for the introduction of Clifford and spinor algebras. This seems to be quite natural because one cannot even write down a wave equation without using a metric, and because, for instance, a simple wave equation in three or more dimensions (e.g., the Klein–Gordan equation) has conformal symmetry.

(B) The relation between the hidden $U(2)$ world and the common real space-time $\mathbb{M}_4$ with tangential space $\mathbb{R}_1^4$ is thought to be more complicated than a simple identification of tangent spaces. The construction should contain the possibility of *identification of indivisible* (in Bohr's sense) $U(2)$ *objects* as observable quanta from the common space-time.

The aim of the paper is, following these concepts, to present algebraic elements for a geometric *ad hoc* construction of a common tangent space $\mathbb{R}_1^4$ from tangent spaces of our $U(2)$ charge model. These elements should give a rigid frame for further differential geometric work and are used for discussions of several quantum concepts, especially the identity problem, and for some new speculations on what a charge is and what a mass is. This aim is also commented on at the end of the next section.

2. IDENTIFICATION OF THE OBJECTS OF A HIDDEN $U(2)$ WORLD

This section is devoted to a sketch of a suggested hidden $U(2)$ world and to a proposal on how their objects can be identified with observable objects in our $\mathbb{M}_4$ world.

The basic construction is a $U(2)$ charge (Donth, 1984) that is thought to be of "infinitely large" size. There are many of them, which are assumed to be "mutually interpenetrating," like the Coulomb fields of different charges in real space.

Three different realizations of the basic $U(2)$ construction can be discussed: leptonic spinors ψ, exchange (or connection) vectors A, and vacuum elements ϕ.

2.1. ψ Individuals

Basic elements of the construction are the so-called ψ individuals

$$\psi: \quad \{U(2), g_w, A_i^{mk}\}, \qquad i=1 \text{ or } 2 \tag{2}$$

They consist of the $U(2)$ manifold $[U(2)=S^3\times S^1]$ with a pseudo-Riemannian bi-invariant (standard) metric, the so-called *wave metric* g_w,

$$g_w: \quad ds^2=-(\sin^2\vartheta_3\, d\varphi_1^2+\cos^2\vartheta_3\, d\varphi_2^2+d\vartheta_3^2)+d\tau^2 \tag{3}$$

written in biharmonic coordinates on $U(2)$,

$$\{\varphi_1, \varphi_2, \vartheta_3, \tau\}, \qquad 0\le\varphi_1, \varphi_2, \tau<2\pi, \qquad 0\le\vartheta_3<\pi/2 \tag{4}$$

and electrodynamic potentials A_i, $i=1$ or 2, in the direction of the S^3 coordinates φ_1 or φ_2,

$$\begin{pmatrix} m+ \\ k\mp \end{pmatrix} := A_1^{mk}=\exp(i\omega\tau-im\varphi_2)\tilde{y}_{mk}(x), \qquad x=\cos\vartheta_3 \tag{5a}$$

$$\overline{\begin{pmatrix} m- \\ k\pm \end{pmatrix}} := A_2^{mk}=\exp(i\omega\tau-im\varphi_1)\tilde{y}_{mk}(\xi), \qquad \xi=\sin\vartheta_3 \tag{5b}$$

The advantages of the coordinates (4) are discussed by Schrödinger (1940). They correspond, in a way, to the Heegaard diagram of genus 1 for S^3 (two symmetric tori with souls $\vartheta_3=0$ and $\vartheta_3=\pi/2$). The potentials (5a), (5b) are periodic solutions of equations (1a), (1b). The coordinates ϑ_3 and τ cannot carry transverse wave potentials: $A_3=0$, $A_4=0$. The y_{mk} functions are harmonic orthogonal polynomials depending on the ϑ_3 coordinate of $S^3\subset U(2)$ [cf. equation (11), below]. $A_i\neq 0$ also applies for $m=0$, $i=1$ or 2.

The term "individual" for $\psi(2)$, and $A(9)$ below, was chosen in order to remind us of the indivisibility or wholeness of quantum phenomena.

Interpreting the individuals as charges, then m is the eigenvalue of the charge, the index $i=1$ or 2 distinguishes positive and negative charges, k is the lepton generation number, for instance

$$\text{electrons } e, \mu, \tau, \ldots: \qquad \begin{pmatrix} - \\ \pm \end{pmatrix}, \begin{pmatrix} - \\ \pm\pm \end{pmatrix}, \begin{pmatrix} - \\ \pm\pm\pm \end{pmatrix}, \ldots \tag{6a}$$

$$\text{neutrinos } \nu_e, \nu_\mu, \nu_\tau, \ldots: \qquad \begin{pmatrix} 0 \\ \pm \end{pmatrix}, \begin{pmatrix} 0 \\ \pm\pm \end{pmatrix}, \begin{pmatrix} 0 \\ \pm\pm\pm \end{pmatrix}, \ldots \tag{6b}$$

B: $\binom{-}{0}$ seems to be a good candidate for the baryon charge, and the "connection" is given by

$$\omega=2k+m \tag{7}$$

Exotics such as

$$E_0^{--}:\ \begin{pmatrix} -\,- \\ 0 \end{pmatrix},\qquad E_1^{--}:\ \begin{pmatrix} -\,- \\ \pm \end{pmatrix},\ldots \tag{8}$$

are also possible.

The two tori of the Heegaard splitting are not connected by sealing their "boundaries," but there is some kind of junction by means of "overlapping" the two conjugate equations (1a), (1b) or (5a), (5b).

The topological aspects of the electrical charges are hidden in the two chained-up tori of the Heegaard splitting.

2.2. *A* Individuals

Consider a world consisting only of nonlocal $U(2)$ objects such as (2). If one wishes to describe a configuration of or an interaction between them, one should have a connection between them.

I suggest we construct a common space-time from elements stemming only from $U(2)$ objects. Thus, we will not construct a space-time that would be independent from the $U(2)$ individuals. That is, the borderline between form (space-time) and contents (physics) will be shifted a little in the direction of physics.

The main idea is that the structure of the common physical space results from a $U(2)$ construction based on an overlap of two conjugate solutions (5a), (5b). Then we would have the possibility to quantize the interaction carriers (A individuals) and to identify them as vector particles. [Taking only the ordinary tangent spaces of the $U(2)$ manifolds would not give such simple possibilities.]

The following construction principle is suggested. An *A individual* consists [in the sense of equations (5a), (5b)] of two conjugate $U(2)$ objects like (2) with a junction $*$ between them:

$$A=\begin{pmatrix} m_1 + \\ v \end{pmatrix} * \begin{pmatrix} m_2 - \\ v \end{pmatrix} = (A_1\,\text{type}) * (A_2\,\text{type}) \tag{9}$$

For instance,

$$\begin{aligned} A&=\begin{pmatrix} 0 \\ v \end{pmatrix} * \begin{pmatrix} \bar{0} \\ v \end{pmatrix}, \qquad & Z^0&=\begin{pmatrix} + \\ v \end{pmatrix} * \begin{pmatrix} - \\ v \end{pmatrix} \\ W^+&=\begin{pmatrix} + \\ v \end{pmatrix} * \begin{pmatrix} \bar{0} \\ v \end{pmatrix}, \qquad & W^-&=\begin{pmatrix} - \\ v \end{pmatrix} * \begin{pmatrix} 0 \\ v \end{pmatrix} \end{aligned} \tag{10}$$

No comprehensive definition for the term junction has been found. But only a preliminary definition is needed for the purpose of the present paper [see Remark 5.1 and equation (76) and Table III below]. The generation number k of the ψ's is replaced by a vacuum susceptibility v in the A's, indicating the presumed importance of the vacuum for the construction of the "connection individuals" A from two parts. Using v instead of k also means the assumption that there is no generation problem with the A's.

2.3. Vacuum Elements ϕ

I refer to the general instability of elementary particles increasing with their growing complexity. This is especially important in the case of $k \to \infty$, which is linked to the concepts of vacuum and of the classical body in quantum experiments. We ask: What could be the decay products of a ψ individual for $k \to \infty$? Let us consider a simple example for the main idea.

The y_{mk} polynomials (5a), (5b) can be written in the form

$$y_{mk}(z) = z^{m/2} \sum_{\kappa=0}^{k} (-1)^{\kappa} \binom{k}{\kappa} \binom{k+m-1}{m+\kappa} z^{\kappa} (1-z)^{k-\kappa} \tag{11}$$

where $z = x^2$ and $1 - z = \zeta = \xi^2 = 1 - x^2$. *Vacuum elements* have been defined as $A_i^{0k'}$ functions with small k' values (and, perhaps, small m' values instead of $m' = 0$?) *being members of a series expansion* of the big A_i^{0k}, $k > (\gg)\ k'$. In the special example of series (11) we have the virtual neutrinos of only one generation,

$$\phi: \quad \{\nu_z = \exp(2i\tau)z,\ \nu_\zeta = \exp(2i\tau')\zeta\} \tag{12}$$

for

$$A_1^{0k} = \cdots + (-1)^{\kappa} \binom{k}{\kappa} \binom{k-1}{\kappa} \nu_z^{\kappa} \nu_\zeta^{k-\kappa} + \cdots \tag{13}$$

with $\tau = \tau'$. They are called virtual because *neither* ν_z^{κ} *nor* $\nu_\zeta^{k-\kappa}$ *nor their product is a "true" individual* in the sense of equations (5a), (5b) for $\kappa, k - \kappa > 1$. The instability is believed to be caused by the large vacuum degeneracy as expressed by large "statistical weights," represented by the binomials in equations (11) or (13). The virtual decays are believed to be the source of the stochastic elements in quantum theories.

2.4. Renormalization

The main idea about renormalization in our frame is that the many vacuum elements must be taken into consideration for an experiment (Donth, 1986). Consider the example (13) again. An experiment M is assumed to be represented by a formula of the structure

$$M = g^{\kappa} A^{\kappa} B^{k-\kappa} \binom{k}{\kappa}^2 \tag{14}$$

The binomial comes from equation (13) for large k; A and B are normalization constants linked with the vacuum elements (12); and g is a coupling constant translating the power of ν_z, for instance, from the $U(2)$ level into the M level. In the limit $k \to \infty$ for fixed

$$y := \lim(\kappa/k) \tag{15}$$

$0 < y < 1$ ($y \neq 1/2$ means a break of vacuum symmetry), one obtains a fixed value of $g_M = g_M(y, A, B)$ for the *actual* g having the following property:

$$\begin{aligned} &M = 0 && \text{for} \quad g < g_M \\ &M \text{ is not defined} && \text{for} \quad g = g_M \\ &M = \infty && \text{for} \quad g > g_M \end{aligned} \tag{16}$$

Therefore, we obtain exactly one value of the coupling constant, $g = g_M$, for a finite experimental value of M.

The numerical value of M is not determined by equation (16). If we assume that the fields ψ and A in $\mathbb{M}_4$ are also influenced by the primary potentials A_i of the ϕ's in the limit $k \to \infty$, then they also become totally indefinite (which is a prerequisite for all renormalizations). Therefore formula (16) has the effect that the numerical values of all physical variables have to be determined anew in the common space-time physics.

Remark 2.1. (General construction principle.) At first sight, only the ψ individuals seem to be reasonable objects from equations (1a), (1b). The other constructions come from the following idea: Individuals with large eigenvalues k (or m) are not stable, but their decay products can be combined anew [A individuals, hadronic individuals as combinations with the baryon charge (Donth, 1986), ..., experiments] and can form in total the vacuum with the nonindependent vacuum elements.

2.5. Comments on the Aim of the Paper

The present paper follows a general $U(2)$ *program* (Donth, 1986; Donth and Lange, 1986) to prove the physical significance of our charge model.

It is not my only objective to recover the well-known spinors anew, but I will also check whether the hidden $U(2)$ structures are consistent with the classical constructions of Dirac and van der Waerden (see, for instance, Dirac, 1958; van der Waerden, 1974) or of Brauer and Weyl (1935), or, in a sense, of Penrose (1968).

The general question of how to derive the structure of the natural space-time from certain physical conditions is not so new. Only some concepts are mentioned: Weyl's ingenious argument for the Pythagorean metric (Weyl, 1923), the Huygens principle (Courant, 1962), and von Weizsäcker's and Bopp's ur-spinors (von Weizsäcker, 1986). The variant pursued here has new aspects, since it starts from electrical charges as the primary concept and uses an explicit model for a hidden $U(2)$ world as being the presumed reality behind quantum theory.

3. MATHEMATICAL SYMBOLS

The following symbols will be used according to the notions of van der Waerden (1967, 1971, 1974) and Dubrovin *et al.* (1986).

Quaternion units: $\{1, l, j, k\}$.

Imaginary unit: i, $i = l$ when related to the quaternion units.

Quaternion coordinates: $\{a, b, c, d\}$, that is,

$$q = a + bl + cj + dk, \qquad q \in \mathbb{H}$$

$\mathbb{R}^4_1$ coordinates: $x^k = \{t, x, y, z\}$, $c = 1$.

$su(2)$ algebra: $\{s^1, s^2, s^3\} = \{s^x, s^y, s^z\}$.

$u(2)$ algebra:

$$\begin{aligned} u(2) &= \{i1, s^1, s^2, s^3\} = i\sigma^k \\ &= \left\{\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right\} \end{aligned} \tag{17}$$

Quaternion units in $\mathbb{C}^2 \sim \mathbb{H}$:

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right\} = \{1, l, j, k\} \tag{18}$$

with a correspondence between the last three matrices of equations (17) and (18) that interchanges the order,

$$(x, y, z) \sim (k, j, l) \tag{19}$$

Then we have

$$\begin{aligned} \mathbb{H} \ni q = x + yj &= \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} \\ \overset{1}{u} = x = a + bi, &\qquad \overset{2}{u} = y = c + di \end{aligned} \tag{20}$$

for the $\mathbb{C}^2$ vectors x and y (not to be confused with the $\mathbb{R}^4_1$ coordinates).

Pauli matrices (including the unit matrix):

$$\sigma^k = \frac{u(2)}{i} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \tag{21}$$

Van der Waerden matrix:

$$C = \begin{pmatrix} c_{11\cdot} & c_{12\cdot} \\ c_{21\cdot} & c_{22\cdot} \end{pmatrix} = \begin{pmatrix} z + t & x - iy \\ x + iy & -z + t \end{pmatrix} = \sum_k \sigma^k x^k \tag{22}$$

$SL(2,\mathbb{C})$ on $\mathbb{C}^2$:

$$g(SL(2,\mathbb{C}))=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\qquad \alpha\delta-\gamma\beta=1 \tag{23}$$

The basis (18) of the spinor space $\mathbb{C}^2$ is distinct from the $u(2)$ algebra (17) in the first matrix, and this difference relates to the decompactification of the $u(2)$ "overlap" by $\mathbb{C}^2$. For the time being one cannot see how $u(2)$ can come into operation. A clue is given by the van der Waerden matrix (22) with a Hermitian basis $\sigma^k=-iu(2)$, which is complementary to $u(2)$ in the sense that the sum of $u(2)$ and $iu(2)$ constitutes the eight units for all complex 2×2 matrices when one considers linear combinations with only real (geometric) coefficients.

Remark 3.1. (Orthogonality and linearity.) Orthogonality is an important concept for the construction of spinor representations. We have to show how the orthogonality $g_{w12}A_1A_2=0$ of our harmonic potentials on $U(2)$ is transferred to orthogonal (conformal) constructions for $\mathbb{R}^4_1$. The potential 1-forms should be linearly transferred from $U(2)$ to $\mathbb{R}^4_1$, otherwise we have no chance of obtaining the superposition principle of quantum theory. No powers of A^2 or A^4 are allowed here, which seem to be unavoidable in usual tensor constructions without orthogonality.

4. VECTOR SPACE $\mathfrak{M}$ [TANGENT SPACE AT $U(2)$]

Consider the tangent space at the manifold $U(2)$. This is a four-dimensional vector space $\mathfrak{M}$. The vectors therein are denoted by u' and their units by u'_i,

$$\mathfrak{M}:\quad \{u'_1, u'_2, u'_3, u'_4\}$$

$$u'=\sum u'_i\beta'_i=u'_1\beta'_1+u'_2\beta'_2+u'_3\beta'_3+u'_4\beta'_4 \tag{24}$$

The unit components will directly be linked with the orthogonal biharmonic coordinates (4) on $U(2)$, $\{\varphi_1,\varphi_2,\vartheta_3,\tau\}$,

$$u'_1\sim\partial/\partial\varphi_1,\qquad u'_2\sim\partial/\partial\varphi_2,\qquad u'_3\sim\partial/\partial\vartheta_3,\qquad u'_4\sim\partial/\partial\tau \tag{25}$$

No serious problems can arise from the coordinate system singularity at $\vartheta_3=0,\ \pi/2$, if the renormalization is connected with an average angle of $\vartheta'_3\neq 0,\ \pi/2$ ($y\approx\sin^2\vartheta'_3$).

The wave metric (3) in $U(2)$ corresponds to a bilinear form Q of the coordinates β'_i on $\mathfrak{M}$,

$$Q\sim g_w \tag{26a}$$

which is assumed to be orthonormal

$$Q(u) = Q(\beta_1', \beta_2', \beta_3', \beta_4') = \sum q_i \beta_i'^2 \tag{26b}$$

Q and g_w have the same signature

$$(---+) \tag{27}$$

that is, we have chosen the unit form

$$q_1 = q_2 = q_3 = -1, \qquad q_4 = +1 \tag{28}$$

There is no loss of generality with the orthonormalization equations (26b) and (28) because the renormalization property (16) allows such a separation between space and field. The construction of $\mathfrak{M}$ does not directly refer to the $U(2)$ topology of the manifold; it could also be the tangent space of another four-dimensional manifold.

5. CLIFFORD ALGEBRA $\mathfrak{G}$ WITH RESPECT TO Q OF $\mathfrak{M}$

Usually, tensor constructions are used for higher spaces, for example spin-tensors, in order to obtain vectors from spinors. Component products are also used in the representation theories. But it is our ultimate aim to construct a connection in the common space that is mediated by the A individuals (9) with a definite geometry. Considering an algebra geometrically as a local representation of a nontrivial geometry such as S^3, then the higher space should be able to contain such an algebraic structure, which at least makes it possible to construct $S^3 \subset U(2)$. First a Clifford algebra $\mathfrak{C}$ over $\mathfrak{M}$ will be chosen because it is constructed by the form Q retaining an essential element of our $U(2)$ individuals, namely the metric (orthogonality concept).

Assuming it is possible to find elements u_i with the properties (α) any u_i linearly depends on u_i' [equation (25), same index] and (β) they can linearly depend on the desired map from $U(2)$ to the common space, then the 16 units of $\mathfrak{C}$ are

$$\begin{gathered} 1 \\ u_1, \quad u_2, \quad u_3, \quad u_4 \\ u_1u_2, \quad u_2u_3, \quad u_3u_1, \quad u_4u_1, \quad u_4u_2, \quad u_4u_3 \\ u_1u_2u_3, \quad u_2u_3u_4, \quad u_3u_4u_1, \quad u_4u_1u_2 \\ u_5 := u_1u_2u_3u_4 \end{gathered} \tag{29}$$

with the following relations between the generating elements:

$$u_iu_i = q_i, \qquad u_iu_j + u_ju_i = 0 \qquad (i<j) \tag{30}$$

In the invariant language, the space constructed is a residue class construction with respect to the metrics. More precisely: Let $\mathfrak{T}$ be the tensor ring and $\mathfrak{J}$ the two-sided ideal which is generated by

$$uu - Q(u) \tag{31}$$

Then $\mathfrak{C}$ is the ring of residue classes $\mathfrak{T}/\mathfrak{J}$. Roughly speaking, all the metric constructions which are equivalent with respect to equation (31) are considered as one element of the algebra units.

Example 5.1. The following statement (van der Waerden, 1971, 1967; Dubrovin *et al.*, 1986) shows explicitly the possibility of bilinear geometric constructions from $\mathfrak{C}$. The second Clifford algebra $\mathfrak{C}_+$ of the ternary quadratic form

$$Q(\beta_1', \beta_2', \beta_3') = q_1\beta_1'^2 + q_2\beta_2'^2 + q_3\beta_3'^2 \tag{32}$$

is the algebra of general quaternions. The space of Hamiltonian quaternions $\mathbb{H}$ is obtained by $q_1 = q_2 = q_3 = -1$. The relation to S^3 is mediated by the quaternion norm $q\bar{q} = 1$, $q^{-1} = \bar{q}$, $q \in \mathbb{H}_1$,

$$\mathbb{H}_1 = S^3 \tag{33}$$

See also Example 7.1 below.

Remark 5.1. [Identity (i).] In order to define the identity problem of the interpretation of the quantum mechanical formalism (see, e.g., Cramer, 1986), we must ask ourselves: What is the wave function ψ, and, more specifically, what is its components? How can one link ψ_R, ψ_L; ψ, ψ_c; ψ, ψ^*; $\psi_\uparrow$, $\psi_\downarrow$; ..., with the particles?

Let us recall once more that it is our aim to include the general construction principle (9) for A individuals into the construction of the common space. But then we must be able to represent single individuals in it, too!

Considering first the A-based construction principle, we can link the following picture (*two-u construction*) with the generating equation (30). *One u comes from the A_1-type part, and the other u comes from the conjugate A_2-type part of equation* (9). This construction can be interpreted as one realization of the junction concept of this equation. We can relate the most nonspecific character of the residue class construction for the common space to the great number of vacuum elements needed for the susceptibility v. This means that the bare units u_i do not contain information from single individuals.

Consider now single ψ (or A) individuals in this space. Their potentials (A_1, A_2) are orthogonal with respect to the general metric (3). Denoting the (map of renormalized) individual potentials in the higher spaces by β_i, then it follows from the orthogonality that all the β-coordinates for the 16 units (29) of the higher space are, at most, linear in β. That means that the individuals are linearly expressed by coordinates (β) in the higher space. More details on this point (spinor construction) are presented in Remark 7.1.

It should again be stressed that, from the very beginning, the bilinear elements (30) are common to both generating parts of equation (9). Because of the high symmetry of equations (30), too, there is no sense in asking whether a (or which) u_i in equation (30) is linked with a given ψ individual. This means that the ψ individual living in constructions with units from equation (30) must principally be able to go into either of the two "u rooms of our new house" that originally correspond to the two conjugate parts of the construction (9). All details of the $U(2)$ individuals, i.e., the behavior of the A_i ($\varphi_1, \varphi_2, \vartheta_3, \tau$) functions, except for the common metric (orthogonality), are extinguished by the residue class construction (21).

Our two-u construction is, as follows from the further development of these ideas (see especially Remark 7.1), the reason for the well-known facts that ψ_R and ψ_L are not different particles but only components of one massive particle, that a common field must be used for electrons ψ and positrons ψ_c in the Dirac four-spinor, and so on. The full identity of the individual particles is therefore not given *a priori* by the ψ field only, but must be determined by the mass and charge construction, which is as yet unknown. This point will be discussed further in Remark 11.5 and in Section 13.

6. SECOND CLIFFORD ALGEBRA $\mathfrak{G}_+$ OF $\mathfrak{G}$

The selection $\mathfrak{C}_+ \subset \mathfrak{C}$ is motivated by the two-u construction of Remark 5.1 that distinguishes even combinations of u. Further, Example 5.1 also calls for a second Clifford algebra, since S^3 elements are needed in our construction $[U(2) = S^3 \times S^1]$.

The eight unit components of $\mathfrak{C}_+$ are denoted by

$$\mathfrak{C}_+ = \begin{Bmatrix} 1, & l, & j, & k, & E, & L, & J, & K \\ 1, & u_1u_2, & u_2u_3, & u_3u_1, & u_5, & u_4u_3, & u_4u_1, & u_4u_2 \end{Bmatrix} \tag{34}$$

Now the definition of the u's [(α) and (β) of Section 5] can be limited to even-u combinations. For instance, one can relate the generating relations (30) to derivations of corresponding scalar products, permitting the application of fine geometric methods of the orthogonality concept.

The multiplication table of $\mathfrak{C}_+$ (column times row) is obtained from equation (30) and is given in Table I.

The following abbreviations will be used. An element of

$$\mathfrak{C}_+ = \{\mathbb{H}, \tilde{\mathbb{H}}\} \tag{35}$$

Table I

	1	l	j	k	E	L	J	K
1	1	l	j	k	E	L	J	K
l	l	-1	$+k$	$-j$	L	$-E$	$+K$	$-J$
j	j	$-k$	-1	$+l$	J	$-K$	$-E$	$+L$
k	k	$+j$	$-l$	-1	K	$+J$	$-L$	$-E$
E	E	L	J	K	-1	$-l$	$-j$	$-k$
L	L	$-E$	$+K$	$-J$	$-l$	$+1$	$-k$	$+j$
J	J	$-K$	$-E$	$+L$	$-j$	$+k$	$+1$	$-l$
K	K	$+J$	$-L$	$-E$	$-k$	$-j$	$+l$	$+1$

is called a *biquaternion*, where

$$\mathbb{H} = \{1, l, j, k\} \tag{36a}$$

is the quaternion field, and where

$$\begin{aligned} \tilde{\mathbb{H}} &= -\mathbb{H}i = -i\mathbb{H} = \{E, L, J, K\} \qquad (i = l) \\ \tilde{\mathbb{H}} &= u_5\mathbb{H} = \mathbb{H}u_5, \qquad u_5 = E, \qquad E^2 = -1 \end{aligned} \tag{36b}$$

From Table I one obtains

$$\mathbb{H} \cdot \mathbb{H} = -\tilde{\mathbb{H}} \cdot \tilde{\mathbb{H}} = \mathbb{H} \tag{37}$$

$$\tilde{\mathbb{H}} \cdot \mathbb{H} = \mathbb{H} \cdot \tilde{\mathbb{H}} = \tilde{\mathbb{H}} \tag{38}$$

In particular, notice that there is no one-to-one correspondence either between $\mathbb{H}$ and the A_1-type or between $\tilde{\mathbb{H}}$ and the A_2-type parts of an A individual. According to Remark 5.1, both parts are well mixed inside both $\mathbb{H}$ and $\tilde{\mathbb{H}}$.

Remark 6.1. Biquaternions are associative and therefore distinct from octonions. The Frobenius-Pontryagin theorem would make it difficult to obtain locally compact and connected fields when we go too far beyond quaternions. The "no go" of a linkage between "internal" and "external" symmetries is avoided by algebraic structures built into the latter spaces from the very beginning. $\{\mathbb{H}, \tilde{\mathbb{H}}\}$ is isomorphic to the general matrix algebra $M(2, \mathbb{C})$, to Imaeda's (1976) biquaternions $\mathbb{Z}$, and to the Clifford algebra $\mathfrak{C}$ of $\mathbb{R}^3$ with the signature $(+++)$, but is not isomorphic to Clifford's biquaternions $\mathbb{H} \oplus \mathbb{H}$, being $\mathfrak{C}$ of $\mathbb{R}^3$ with $(---)$; cf. also the classification by Salingaros (1981).

7. SPINOR SPACES. CONSTRUCTION OF TWO COMPLEX SPACES $\mathbb{C}^2$ AND $\dot{\mathbb{C}}^2$

As is well known, $\mathbb{H} \sim \mathbb{C}^2$. There are two $\mathbb{C}^2$ spaces in the van der Waerden construction of spinor spaces, $\mathbb{C}^2$ and $\dot{\mathbb{C}}^2$, where $\dot{\mathbb{C}}^2$ is defined as a space that is transformed, under a given $SL(2, \mathbb{C})$ group on $\mathbb{C}^2$, by complex conjugate elements. We ask whether one can construct a $\tilde{\mathbb{C}}^2$ space from $\tilde{\mathbb{H}}$ with the property $\tilde{\mathbb{C}}^2 = \dot{\mathbb{C}}^2$.

The construction of $\mathbb{C}^2$ from $\mathbb{H}$ is well known:

$$\begin{aligned} q &= a + bl + cj + dk \in \mathbb{H} \\ &= x + yj = x + j\bar{y} \end{aligned} \tag{39}$$

$$\mathbb{C}^2 = \{x, y\}, \qquad x = a + bi, \qquad y = c + di \\ \bar{y} = c - di \qquad (i = l) \tag{40}$$

The multiplication of two quaternions remains on $\mathbb{C}^2$ with the components (x', y')

$$\begin{aligned} (x + yj)(u + vj) &= (x + yj)(u + j\bar{v}) \\ &= \underbrace{(xu - y\bar{v})}_{x'} + \underbrace{(xv + y\bar{u})}_{y'}j \end{aligned} \tag{41}$$

The construction of $\tilde{\mathbb{C}}^2$ from $\tilde{\mathbb{H}}$ is analogous,

$$\tilde{q} = \tilde{a}E + \tilde{b}L + \tilde{c}J + \tilde{d}K \in \tilde{\mathbb{H}} \tag{42}$$

$$\tilde{q} = \tilde{x} - \tilde{y}j, \qquad \tilde{x} = \tilde{b} - \tilde{a}i, \qquad \tilde{y} = \tilde{d} - \tilde{c}i \tag{43}$$

$\tilde{\mathbb{C}}^2 := \{\tilde{x}, \tilde{y}\}$ is also a $\mathbb{C}^2$ space with a stable multiplication with a similar component structure as before, and

$$(\tilde{x} - \tilde{y}j)(\tilde{u} - \tilde{v}j) = (\tilde{x}\tilde{u} - \tilde{y}\bar{\tilde{v}}) - (\tilde{x}\tilde{v} - \tilde{y}\bar{\tilde{u}})j \tag{44}$$

Thus,

$$\mathbb{H} = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}, \qquad \tilde{\mathbb{H}} = \begin{pmatrix} \tilde{b} - \tilde{a}i & \tilde{d} - \tilde{c}i \\ \tilde{d} + \tilde{c}i & -\tilde{b} - \tilde{a}i \end{pmatrix} \tag{45a}$$

From these formulas one can see that $\tilde{\mathbb{C}}^2$ is conjugate to $\mathbb{C}^2$ with respect to

$$j \to -j, \qquad i \to -i, \qquad \text{“}a \leftrightarrow b, \quad c \leftrightarrow d\text{”} \tag{45b}$$

Remark 7.1. (Transformations in $\mathbb{C}^2$ and $\dot{\mathbb{C}}^2$.) Two questions will be discussed in this quite lengthy remark. (1) Does $\tilde{\mathbb{C}}^2 = \dot{\mathbb{C}}^2$? (2) What kind of transformation occurs?

Generally speaking, we suggested that the coordinates (β) of the physical vectors on $\mathfrak{C}_+$ are related to the potential 1-forms (5a), (5b), A_1 and A_2, on the individuals. In our nonlocal interpretation of the component solutions (5a), (5b) as electrical charges, "one" $U(2)$ bearing both A_1 and A_2 would correspond to the case of two ψ individuals with different charge signs. Moreover, two individuals of + charge and one individual of (++) charge are also different things [because all y_{mk} functions are different from one another for different (m, k)]. According to equation (3), only the potentials (5) carry the information (i, m, k)—our knowledge—about the kind of ψ individuals. Two conclusions can be drawn from these statements.

Concerning question (1). The relation between the $U(2)$ level and the $\mathfrak{C}_+$ level will be denoted by $\sim$. According to Remark 5.1, the coordinates β_i, which are linked to the corresponding u_i (same index i), can only be related to the corresponding $U(2)$ potentials A_i (same index i). This implies that there are no variable β's when there are no potentials in the corresponding directions. This is expressed by $\beta_3 \sim 1$, $\beta_4 \sim 1$. Since φ_1 and φ_2 can carry potentials, variable β's are possible in these directions, $\beta_1 \sim A_1$, $\beta_2 \sim A_2$. The term "variable" means that, although the β's cannot yet ultimately be defined [because we do not have a complete definition of the $u_i u_j$ combinations (30) and because they are uncertain in the sense of the experiment, equation (16)], they can be renormalized and they can be varied by transformations. For the ψ's, either β_1 is the variable (for ψ_1, that is, a ψ individual bearing A_1), or β_2 is the variable (for the anti-individual ψ_2). For the A's the orthogonality must be taken into consideration. This means $\beta_1\beta_2 \sim 1$ for the A's (no variable information for strong orthogonality).

Summing up, we have

$$\beta_1 \sim A_1, \qquad \beta_2 \sim A_2, \qquad \beta_3 \sim 1, \qquad \beta_4 \sim 1$$

$$\beta_1\beta_2 \sim \begin{cases} \beta_1 & \text{for } \psi_1 \\ \beta_2\cdot & \text{for } \psi_2 \\ 1 & \text{for } A \end{cases} \tag{46}$$

All the variable β's are listed in Table II for the physical coordinates of the individuals.

The transition from the formal (tensorial) attachment in the third column to the individual attachments for ψ and A is made using formulas (46). Table II shows that the $U(2)$ to $(\mathbb{H}, \tilde{\mathbb{H}})$ potential transfer $A_i \to \beta_i$ is linear in all nonformal cases.

The "primary anisotropy" of the physical β component distribution over $\mathbb{H}$ and $\tilde{\mathbb{H}}$ in Table II is not a serious problem. According to the next-to-last paragraph of Remark 5.1, all directions are principally accessible and we can make a rotation of the frame in order to get other components in the $\mathbb{H}$'s.

Table II

Basis	q or $\tilde{q}$	Formal	Physical		
			ψ_1	ψ_2	A
$1=1$	a	1	1	1	1
$l=u_1u_2$	b	$\beta_1\beta_2$	β_1	β_2	1
$j=u_2u_3$	c	$\beta_2\beta_3$	1	β_2	β_2
$k=u_3u_1$	d	$\beta_1\beta_3$	β_1	1	β_1
$E=u_5$	$\tilde{a}$	$\beta_1\beta_2\beta_3\beta_4$	β_1	β_2	1
$L=u_4u_3$	$\tilde{b}$	$\beta_3\beta_4$	1	1	1
$J=u_4u_1$	$\tilde{c}$	$\beta_1\beta_4$	β_1	1	β_1
$K=u_4u_2$	$\tilde{d}$	$\beta_2\beta_4$	1	β_2	β_2

As in Table II, we will connect the term "physical" with the β variables in the sense of equation (46). This term will also be applied to the corresponding variables φ_1, φ_2. The other variables, τ and ϑ_3, will occasionally be called "hidden," although a sharp separation is not possible, since all four u's are well mixed in the two-u construction.

Transformations are only interesting for physical variables. Let us take the biquaternion coordinates from Table II that are equivalent ($\approx$) in the "primary" β pattern for ψ and A as a whole. Then

$$a \approx \tilde{b}, \qquad b \approx \tilde{a} \tag{47}$$

$$c \approx \tilde{d}, \qquad d \approx \tilde{c} \tag{48}$$

Comparing with the formulas (45a), (45b) we see that $(\tilde{x}, \tilde{y}) \in \tilde{\mathbb{C}}^2$ is transformed by complex conjugate parameters of a $(x, y) \in \mathbb{C}^2$ transformation, which means $i \to -i$, or

$$\tilde{\mathbb{C}}^2 = \dot{\mathbb{C}}^2 \tag{49}$$

Concerning question (2). We must now ask: What is a spinor transformation in the "pull back" to the $u(2)$ level of the individuals? (Spinors are of course vectors in $\mathbb{C}^2$ and $\dot{\mathbb{C}}^2$.)

Only $\{u_1', u_2'\} = \mathbb{R}^2 \subset \mathfrak{M}$ will be considered, because u_3 and u_4 do not carry information. Since we do not have any characteristic length in the $U(2)$ manifold—there is no finite $U(2)$ or S^3 "radius," and all the variables are angles—we can image (cf. Section 2) that the $u(2)$'s are "infinitely large," like the range of Coulomb fields, and that their "size" can be varied by scaling. The simplest transformation at the $U(2)$ level is a scale comparison of the $\mathbb{R}^2$'s for two individuals. Provided that the orthogonal structure is conserved in the comparison, the transformation is conformal. Therefore, we shall consider the simple conformal transformations of $\mathbb{R}^2$—that is, Liouville's conformal group on $\mathbb{R}^2$ (Weyl, 1923; Dubrovin *et al.*, 1986, §§ 15, 24). This group is isomorphous to $SL(2, \mathbb{C})$ for the spinors,

$$SL(2,\mathbb{C}) = \begin{cases} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & \text{for} \quad \mathbb{C}^2 \\ \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} & \text{for} \quad \dot{\mathbb{C}}^2 \end{cases} \tag{50}$$

$$\alpha\delta - \beta\gamma = 1$$

Remark 7.2. [Identity (ii).] Now we can discuss the physical contents of the spinors. There is no spinor without a ψ individual. Regarding the information about the ψ individuals—our knowledge of the leptons—we must consider the physical contents of the β's. According to equation (46) and Table II, the latter are linked with the electromagnetic potentials A_i on $U(2)$, irrespective of renormalization. Generalizing slightly, we reach an interpretation resembling some early ideas of Schrödinger and de Broglie:

The wave function ψ is, considering its physical contents, the potential 1-form (5a), (5b) *of the $U(2)$ individuals* (2) *realized on the spinor space.*

The β's and therefore the wave functions carry all the information about the ψ individuals: $i = 1$ or 2, m, and k. They are principally complex because the potentials (5a), (5b) have complex exponentials. The latter have $U(1)$ topology and carry the integer m, the electric charge. The transfer $A_i \to \psi$ is linear (superposition principle).

For the time being, only tangent spaces have been used in the construction. No points have yet been constructed. This question (uncertainty) will be further discussed in Section 13.

8. LIE ALGEBRA $u(2)$. POSSIBILITY FOR IDENTIFICATION

According to Remark 5.1 and Table II, a massive particle cannot *a priori* be attached to only one spinor of $\mathbb{C}^2$ or $\dot{\mathbb{C}}^2$. In spite of this identity problem, it must be possible to identify the individuals as a whole, as one indivisible quantum in an experiment of normal size in the common space. Occasionally, this situation will simply be expressed by "as studied from $\mathbb{R}^4_1$, or $\mathbb{M}_4$." In particular, taking the topology as an indication for the wholeness, we should be able to detect their $U(2)$ topology more or less by local and tangent elements.

This is why symmetry is so important in quantum physics: We can conclude from local elements to the individual wholeness only if the individuals have symmetries and if their symmetries are contained in their local elements.

This means that the local contact between one individual and the common space is not made by points but by Lie brackets. Here are two examples: First, according to the Lie theorem, the whole, simply connected group manifold can be obtained from the Lie algebra. Second, according to the Frobenius theorem, a foliation is obtained when the vectors and their Lie derivatives lie exactly on the tangential spaces. Since the Lie brackets principally cover a neighborhood of a point (i.e., more than just a point), the $U(2)$ individuals are principally realized nonlocally in a common space (uncertainty principle).

Therefore, we have to consider a matrix basis that is tangent to $U(2)$ when individuals are to be identified as "particles" or "quanta,"

$$u(2) = \{i\sigma^k\} = \text{equation (17)}$$

In our model the physical elements for the realization of ψ individuals are now

$$\psi: \begin{cases} 2\times 2 \text{ matrices } u(2) = \text{tangent objects as} \\ \quad \text{representatives of the individuals} \\ \{\psi_R, \psi_L\} = \text{spinors} = \text{classical vectors of} \\ \quad \mathbb{C}^2, \dot{\mathbb{C}}^2 \text{ as representatives of their potentials} \end{cases} \tag{51}$$

We will speak about *tangent objects* when we wish to identify $U(2)$ individuals as studied from the common space using $u(2)$, or more generally, when algebraic aspects of ψ (or A) come into play. On the other hand, the conception of vectors tends toward wave functions and space connections. In quantum field theories the two aspects are mutually related, roughly speaking, by Fourier transformations (Weinberg theorem).

The difference between the two aspects of equation (42) is also discussed in Remark 11.4 below. The basis of $\mathbb{C}^2$, equation $(18) \neq u(2)$, corresponds to a noncompact overlapping of this algebra: The matrices (18) have the same commutators as $u(2)$, but they are not unitary.

Remark 8.1. Equation (51), where the two aspects are merely put side by side, is, of course, not the final formulation for ψ in a quantized gauge theory. But it is sufficient for the construction of a reasonable tangent space $\mathbb{R}_1^4$ and for proving the algebraic transformation properties of the tangent objects, which is covered in the next section. It also opens the way to gauge theories where, roughly speaking, both aspects are tightly connected (see Remarks 11.1, 11.6, and the first paragraph of Section 13, below).

Remark 8.2. The utility of the 2×2 matrices σ^k (21) for electrodynamics is well documented by Baylis (1980) and Baylis and Jones (1988), but without mention of their relation to $u(2)$.

9. MATRIX ALGEBRA $M(2, \mathbb{C})$

Now we try to find an algebraic description of the tangent objects, as studied from the common space.

For the sake of simplicity we shall first only consider one spinor space, e.g., $\mathbb{C}^2$. That is, we ignore the identity (i) problem of Remark 5.2 and only consider rotations in $\mathbb{R}^3$. The complete matrix algebra $M(2, \mathbb{C})$ has eight real dimensions. One can choose the anti-Hermitian matrices $u(2) = \{i\sigma^k\}$, equation (6), plus the Hermitian matrices $\{\sigma^k\}$, equation (10), as the eight units. From a geometric point of view, the former are the complement $\complement$ of the latter: both can characterize the geometry of $u(2)$ in the same manner. Therefore, it is of no importance which of them is selected as the possibility for particle identification. We choose the Hermitian matrices

$$\{\sigma^k\} = \complement\ u(2) \tag{52}$$

because of their advantages in providing a quantum mechanical description (real eigenvalues as representatives of experiments).

Matrix algebra is used to describe the ψ transformations in the sense of tangent objects living in the common tangent space $\mathbb{R}_1^4$ (as constructed in the next section). These transformations are different from the transformations (50) that are related to the space $\mathbb{C}^2$ itself or to classical, nonspecific vectors therein [see the comments for equation (51)]. The algebra interchanges the ψ components.

The peculiarities of the tangent object transformations can be demonstrated by (what we term here) the *matrix theorem* (e.g., Brauer and Weyl, 1935, van der Waerden, 1967). For our purpose it can be formulated as follows: Let u' be a vector in a tangent space $\mathfrak{M}'$ with a quadratic form Q'. Then an orthogonal transformation A can be written as

$$Au' = su's^{-1} \tag{53}$$

where $s \in \mathfrak{C}_+$ transforms $\mathfrak{M}'$ into $\mathfrak{M}'$,

$$s\mathfrak{M}'s^{-1} = \mathfrak{M}' \tag{54}$$

In our case $s \in M(2, \mathbb{C})$. Examples are given below.

The vector transformation (50) is linear with the coefficients, whereas the tangent object transformation (53) applies a bilinear, adjugate treatment.

The relation to the identity problem of Remark 5.2 is given by the following physical interpretation: A transformation of tangent objects must make it possible to find the same particle after the situation has been changed (but, unfortunately, it does not guarantee this in $\mathbb{C}^4$; see Remark 11.6).

Remark 9.1. The statements "$u(2)$" and "Hermitian" are equivalent in our treatment (52): in our space construction $u(2)$ is hidden by the concept of Hermitian operators. Conversely, it seems that it is just the $u(2)$ algebra that marks out quantum theory with its Hermitian operators [provided all this can be developed from spinor concepts by properly defined space connections which allow the momentum to be extracted from the angular momentum, and likewise the angular momentum from spinors (Weyl-Heisenberg group?); see Section 13 for a further discussion of this].

Example 9.1. Construction of and transformation in $\mathbb{R}^3$.

(a) (Dubrovin *et al.*, 1986). Consider the (matrix) algebra (20) of the quaternions,

$$\mathbb{H}: \quad q = a + bl + cj + dk \tag{55}$$

The Euclidean space $\mathbb{R}^3$ is determined by the subset of "imaginary quaternions" $\mathbb{H}_0$ defined by $a = 0$,

$$q_0 = x \in \mathbb{H}_0 = \mathbb{R}^3: \quad x = bl + cj + dk \tag{56}$$

with the metric

$$g_0 = -q_0^2 = b^2 + c^2 + d^2 \tag{57}$$

The sphere $S^3 = SU(2)$ is determined by the subset of the normalized quaternions

$$\mathbb{H}_1: \quad q_1 = a + bl + cj + dk \quad \text{with} \quad q_1\bar{q}_1 = 1 \tag{58}$$

which means

$$a^2 + b^2 + c^2 + d^2 = 1 \tag{59}$$

Thus, an orthogonal transformation (rotation α_q) of $\mathbb{R}^3$ $(=\mathfrak{M}')$ is given by the matrix theorem

$$\alpha_q: \quad x \to q_1 x q_1^{-1} \tag{60}$$

(b) In terms of the transformation (50), the following trick is used for translating the vectors from the space $\mathbb{C}^2$ onto the *tangent objects* in $\mathbb{R}^3$ (van der Waerden, 1974). Let $(\overset{1}{u}, \overset{2}{u})$ be the basis of $\mathbb{C}^2$. Then the special form

$$(a_1\overset{1}{u} + a_2\overset{2}{u})(b_1\overset{1}{u} + b_2\overset{2}{u}) \equiv \overset{1}{u}'\overset{2}{u}' \tag{61}$$

is considered to be the product of two *transformed* vectors, one in the direction 1 $(\overset{1}{u} \to \overset{1}{u}')$, and the other in the direction 2 $(\overset{2}{u} \to \overset{2}{u}')$. (These vectors can be thought of as stemming from the primary anisotropy of Table II.) Then the general bilinear form

$$c_0(\overset{1}{u})^2 + c_1\overset{1}{u}\overset{2}{u} + c_2(\overset{2}{u})^2 \tag{62}$$

with the matrix C (22) for $t = 0$ is thought of as being transformed by

$$SU(2): \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1 \tag{63}$$

where the tangential object structure is constructed from the transformed vectors by

$$b_1 = -a_2^*, \qquad b_2 = a_1^* \tag{64}$$

The right-hand side of the matrix theorem is then

$$\begin{pmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{pmatrix} \begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix} \begin{pmatrix} a_1^* & -a_2 \\ a_2^* & a_1 \end{pmatrix} \tag{65}$$

which leaves $x^2+y^2+z^2$ invariant.

Example 9.2. A well-known example for the matrix theorem is the transformation of the Dirac matrices γ_μ as "tangent vectors" in $\mathbb{R}_1^4$ (see, e.g., Fermi 1954).

Vectors in $\mathbb{R}_1^4$:

$$x_\mu \to x'_\mu = \sum a_{\mu\nu} x_\nu, \qquad A = \{a_{\mu\nu}\}, \qquad \bar{A}A = 1 \tag{66}$$

Spinors:

$$\psi \to \psi' = T^{-1}\psi \tag{67}$$

Matrix theorem:

$$a_{\mu\nu}\gamma_\nu = T\gamma_\mu T^{-1} \tag{68}$$

Second Clifford algebra:

$$\begin{gathered} T = 1 - \sum (1/4)\varepsilon_{\mu\nu}\gamma_\mu\gamma_\nu \\ a_{\mu\nu} = 1 + \varepsilon_{\mu\nu}, \qquad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \qquad |\varepsilon| \ll 1 \end{gathered} \tag{69}$$

10. DIRAC'S CLIFFORD ALGEBRA

The program of Sections 8 and 9 leads, of course, to Dirac's Clifford algebra. Therefore we can confine ourselves to some comments. Only two aspects will be commented upon: the construction of the algebra—a residue classs construction in a sense similar to that of Section 5—and the fact that the γ matrices can be considered as objects ("vectors") in $\mathbb{R}_1^4$.

The space doubling

$$\mathbb{C}^4 = \mathbb{C}^2 \oplus \dot{\mathbb{C}}^2 \tag{70}$$

results directly from the fact that we have two relatively independent but algebraically similar $\mathbb{C}^2$ spaces with the properties (37), (38), and Table I. Then we arrive at the matrix algebra $M(4, \mathbb{C})$.

1. One could ask why a residue class construction is needed again. It is for the same reason as in Sections 5 and 6. There we constructed the basis for the common space; here we have to construct the basis for the individuals (tangent objects). The individuals are to be released from all details except the algebraic indications for their topology and the metric (orthogonality concept). The former lead, of course, to the matrix representation of the Dirac γ's, the latter to their Clifford algebra structure withe the same Q form (31) as before.

2. Vectors of $\mathbb{R}_1^4$ can be constructed as [and only as, see Brauer and Weyl (1935)]

$$\text{vector} = \bar{\psi}\gamma^{\mu}\psi \tag{71}$$

and it is also possible to construct a wave equation in $\mathbb{M}_4$ (which has additionally an affine connection) with the aid of scalars,

$$\text{kinetic term} = \gamma^{\mu}\,\partial_{\mu}\psi \tag{72}$$

Both $\mathbb{C}^2$ spaces have four real dimensions and are relatively algebraically independent. One finds that the vectors (71) and therefore the kinetic term (72) can be separately defined and transformed in each of the two subspaces $\mathbb{C}^2, \dot{\mathbb{C}}^2$. According to equation (52), the units for the $U(2)$ ψ tangent objects in these subspaces are the σ^k of equation (21). Thus, we arrive at

$$\bar{\psi}\sigma^k\psi \quad \text{and} \quad \sigma^k\partial_k\psi \tag{73}$$

from which one obtains the generators of the Dirac algebra,

$$1, \gamma^0, \gamma^1, \gamma^2, \gamma^3 \tag{74}$$

in the van der Waerden half-spinor representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{75}$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$. The minus sign comes from the difference between $\mathbb{C}^2$ and $\dot{\mathbb{C}}^2$ and will be explained in Remark 10.3. The $\mathfrak{C}_+$ character of the orthogonal transformation has been demonstrated in Example 9.2.

These comments elucidate the hidden $U(2)$ behind the Dirac algebra. $U(2)$ is completely hidden because of the well-known theorem that the algebra over $\mathbb{C}$ with the generators (74) and the defining relations (28) and (30) is isomorphous to the *general* matrix algebra $M(4, \mathbb{C})$.

11. CONSTRUCTION OF THE MINKOWSKI TANGENTIAL SPACE $\mathbb{R}_1^4$

We would like to have a construction for $\mathbb{R}_1^4$ similar to that for $\mathbb{R}^3$ according to Example 9.1. A direct generalization, however, confronts us with difficulties, for instance, that $\mathbb{H} \neq \mathbb{R}_1^4$, and that we have eight coordinates in the biquaternions (35). We start with an *ad hoc* definition.

A biquaternion $\{q, \tilde{q}\}$ with the coordinates $\{a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ is called a *biquaternion with junction* $\{q, q_c\}$ when

$$\tilde{a} = -a, \qquad \tilde{b} = b, \qquad \tilde{c} = c, \qquad \tilde{d} = d \tag{76}$$

Identification (76) closely links $\mathbb{C}^2$ and $\dot{\mathbb{C}}^2$, or, which is the same, $\mathbb{H}$ and $\tilde{\mathbb{H}}$. According to equations (34), (36a), and (36b), the relation to the junction as used for the two-u construction (30) is given by the binary scheme in Fig. 1. The two-u construction of the one-to-another "interpenetrating" nonlocal $U(2)$ objects implies that, mediated by the large number of vacuum elements, all u_i directions can be "contacted" or "compared" or "overlapped" with one another (this will be expressed by the term "freely rotable"). Equation (76) shows that the coordinates of complementary $u_i u_j$ products (e.g., $u_1 u_2$ and $u_3 u_4$) are equal, which property represents some kind of isotropy and justifies the term "free." The dimension of the space is reduced by equation (76) from 8 to 4, which is of some importance for an electromagnetic correspondence between $U(2)$ and $\mathbb{R}_1^4$ (cf. Remark 12.1 below). Equation (76) is the general basis for bilinear ψ constructions in the common space.

We can then put

$$\mathbb{R}_1^4 = \{q, q_c\} = (\mathbb{H}, \mathbb{H}_c) \tag{77}$$

The metric is constructed analogously to Example 9.1,

$$g = \tfrac{1}{2}(q^2 - q_c^2) = a^2 - b^2 - c^2 - d^2 \tag{78}$$

$[= a^2 - g_0^2$, where g_0 is taken from equation (57)]. The representation of g as a difference, $q^2 - q_c^2$, expresses the degree of independence and equivalence of the two spinor spaces $\mathbb{C}^2$ $(= \mathbb{H})$ and $\dot{\mathbb{C}}^2$ $(= \tilde{\mathbb{H}})$. The proof of

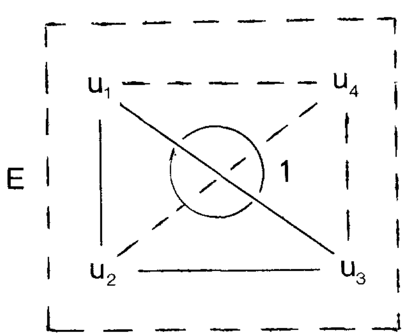

Fig. 1. (— — —) $\tilde{\mathbb{H}}$, $\dot{\mathbb{C}}^2$, H_L, L, (———) $\mathbb{H}$, $\mathbb{C}^2$, H_R, R.

equation (78) shows, however, that the metric (78) is not trivial because all the three objects

$$g, q^2, -q_c^2 \tag{79}$$

are different from one another.

The (physical) coordinates of the tangent Minkowski space $\mathbb{R}_1^4$, $\{t, x, y, z\}$, are listed in Table III (see also Table II).

The common time t, for instance, is constructed by identifying the a coordinates on $\mathbb{C}^2$ and $\dot{\mathbb{C}}^2$, where a and $\tilde{a}$ alone do not have the meaning of a time. The concept of a common time does not come until it is studied in $\mathbb{R}_1^4$. This point is discussed further in Remark 11.4.

Consider the ψ's. The information transfer of the ψ individual potentials A_i, equations (2), (5a) and (5b), via Table II to β_1 and β_2 in Table III, shows that the "empty" (and also the "primary") space is completely isotropic with respect to the information about the ψ's. That is, the material for the wave function in $\mathbb{M}_4$ does not bear any direct trace of the $U(2)$ structure. Therefore it can be formed entirely anew by Schrödinger or Dirac or Poincaré group operators, can suffer new boundary conditions, can get new symmetries, and so on.

Consider the A realization. Table III shows a strict transversality $(A_z \sim 1)$ resulting from the A_i orthogonality on $U(2)$: $\beta_1\beta_2 \sim 1$ according to equation (46).

The β linearity of ψ represents a spinor realization in $\mathbb{R}_1^4$, and the β "bilinearity" of A represents a linear vector representation—linear because the identifying equation (76) is not a multiplication. The vector basis of our $\mathbb{R}_1^4$ (final column of Table III) is also "isotropic" in a sense: All four u's are contained in each of the four $\mathbb{R}_1^4$ direction units. [As also shown by this column, the two spaces R (or $\mathbb{C}^2$) and L (or $\dot{\mathbb{C}}^2$) are different in this aspect. This will be further discussed in Remark 11.3 and Example 13.1.] The $\mathbb{R}_1^4$ directions can be distinguished by their u arrangements. There is a fundamental difference between the time $(1, u_5)$ and the space directions; only the latter contain the generating combinations of equation (30).

Table III

$\mathbb{R}_1^4$	$\{q, q_c\}^a$	ψ_1	ψ_2	A	Units from $\mathfrak{C}_+$ $(R, L)^a$
t	$(a, \tilde{a})$	β_1	β_2	1	$(1, u_1u_2u_3u_4)$
z	$(b, \tilde{b})$	β_1	β_2	1	(u_1u_2, u_4u_3)
y	$(c, \tilde{c})$	β_1	β_2	(β_1, β_2)	(u_2u_3, u_4u_1)
x	$(d, \tilde{d})$	β_1	β_2	(β_1, β_2)	(u_3u_1, u_4u_2)

[a] Note carefully equation (76) and Fig. 1.

Remark 11.1. [Identity (iii).] Table III shows that the coordinates x, y, z, t are physically of the same nature as the coordinates a, b, c, d of $\mathbb{C}^2$ and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ of $\dot{\mathbb{C}}^2$. From comparing with Table II, one can see that the coordinates of $\mathbb{R}_1^4$ and the information about the A individuals (carrying m_1 and m_2) are also of the same physical nature, irrespective of renormalization: they are covariant coordinates. Taking a "slight" generalization from $\mathbb{R}_1^4$ to $\mathbb{M}_4$ into account, one then obtains the following interpretation:

The forms $A = A_\mu\, dx^\mu$ (which can locally define a gauge connection) are, considering their physical contents, the two potential forms on the $U(2)$ individuals (9) that are linearly realized on the Minkowski space.

Just as in the ψ equation (51), we can also expect to find two aspects in A—the covector aspect discussed above and the algebraic aspect. The latter comes into play in the process leading from the old to the gauge field theory. The fields A_μ^a then take values in the Lie algebras $u(1)$ and $su(2)$, which corresponds to the concept of tangent objects as defined in Section 8 (cf. also Remark 11.3, second paragraph). The matrix theorem calls for the use of an adjugate representation for A. Naturally, all this could be better discussed when the junction is completed by a space connection ($\mathbb{R}_1^4 \to \mathbb{M}_4$).

To sum up: we obtained a physical equivalence of the electromagnetic potentials on $U(2)$ and $\mathbb{R}_1^4$ (*correspondence principle*).

Remark 11.2. [Equivalence between our and the van der Waerden construction with the aid of his C tensor (22).] Van der Waerden (1974) (who reached his results back in the late 1920s) constructed $\mathbb{R}_1^4$ as follows. The spinor units are denoted by

$$\overset{1}{u}, \overset{2}{u} \in \mathbb{C}^2, \qquad \overset{1\cdot}{u}, \overset{2\cdot}{u} \in \dot{\mathbb{C}}^2 \tag{80}$$

They are transformed according to equation (50). The complex matrix

$$C = \begin{pmatrix} c_{11\cdot} & c_{12\cdot} \\ c_{21\cdot} & c_{22\cdot} \end{pmatrix} = H + \tilde{H} \tag{81}$$

can always be written as the sum of a Hermitian (H) with an anti-Hermitian ($\tilde{\text{H}}$) part. The space of bilinear forms

$$c_{11\cdot}\, \overset{1\,1\cdot}{uu} + c_{12\cdot}\, \overset{1\,2\cdot}{uu} + c_{21\cdot}\, \overset{2\,1\cdot}{uu} + c_{22\cdot}\, \overset{2\,2\cdot}{uu} \tag{82}$$

is transformed onto itself by the transformation (50) leaving invariant the determinant

$$D_g = c_{11\cdot}\, c_{22\cdot} - c_{12\cdot}\, c_{21\cdot}. \tag{83}$$

If, according to van der Waerden, C is equal to its Hermitian part,

$$C \equiv H = \begin{pmatrix} z+t & x-iy \\ x+iy & -z+t \end{pmatrix} = (22) \tag{84}$$

then the invariant is

$$D = t^2 - z^2 - x^2 - y^2 = inv \tag{85}$$

But if C is made equal to its anti-Hermitian part [$u(2)$ basis according to equation (19)],

$$C \equiv \tilde{H} = \begin{pmatrix} it'+iz' & y'+ix' \\ -y'+ix' & it'-iz' \end{pmatrix} \tag{86}$$

then the invariant is

$$\tilde{D} = -t'^2 + z'^2 + x'^2 + y'^2 = -D \tag{87}$$

Comparing with equation (78), we can see that $g = D = -\tilde{D}$, or formally $= \frac{1}{2}(D - \tilde{D})$, with $a = t' = t$, $b = z' = z$, $c = y' = y$, $d = x' = x$. The geometric content of this construction is hidden in the tangent object units of H [$-iu(2)$] and $\tilde{H}$ [$u(2)$]; cf. Sections 8–10. [Compare, e.g., the expression $\sum \sigma^\mu A_\mu$ for A with equations (71) and (72) for ψ.]

The equivalence between both constructions is given by a relationship between $H, \tilde{H}$ and the biquaternion parts $\mathbb{H}, \mathbb{H}_i$. Neither H nor $\tilde{H}$ is a quaternion representation (20). The relation is achieved by Hamilton's complex quaternions,

$$\begin{aligned} (t, z, y, x) &= (a, ib, ic, id) \\ q &= a + bl + cj + dk \in \mathbb{H} \end{aligned} \tag{88}$$

and

$$\begin{aligned} (t', z', y', x') &= (a, -ib, -ic, -id) \\ q_c &= +\tilde{a}E + \tilde{b}L + \tilde{c}J + \tilde{d}K \in \tilde{\mathbb{H}} \\ &= -aE + bL + cJ + dK \in \mathbb{H}_c \end{aligned} \tag{89}$$

$\{q, q_c\}$ is a biquaternion with junction according to definition (76). Proving equation (89), one must observe equations (45a), (45b) that exchange the roles of a, b and of c, d. The determinant D becomes a quaternion norm,

$$D = q\bar{q} \tag{90}$$

One also obtains $\tilde{D} = q_c\bar{q}_c$ using an analogous definition of $\bar{q}_c$. The property (79) is hidden by the additional imaginary units i in equations (88) and (89). Thus

$$H(a, b, c, d) \equiv H_R \triangleq \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \in \mathbb{H} \tag{91}$$

$$\tilde{H}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \equiv H_L \triangleq \begin{pmatrix} b+ai & d-ci \\ d+ci & -b+ai \end{pmatrix} \in \mathbb{H}_c \tag{92}$$

where the symbol $\triangleq$ denotes the equivalence desired between $u(2)$ A tangent objects H and vector spaces $\mathbb{H}$.

Remark 11.3 (Parity, PCT.) Comparing equations (88) and (89), we can see that a different parity (R, L) can be attached to the spinors from $\mathbb{C}^2$ or $\dot{\mathbb{C}}^2$, respectively, e.g.,

$$\psi_R \in \mathbb{C}^2, \qquad \psi_L \in \dot{\mathbb{C}}^2 \tag{93}$$

This notation has been used a few times before in this paper. Conditions (93) also explain the minus sign in the construction (75) of the γ matrices.

One can see from the final column of Table III that, in spite of the β isotropy of $\mathbb{R}^4_1$, the spaces $R = \mathbb{C}^2$ and $L = \dot{\mathbb{C}}^2$ are quite distinct with respect to the u_iu_j contents and arrangements! The construction of H_R and H_L according to equations (84) and (86) suggests that we can, at least partially, define A tangent objects for R and L separately.

It is difficult to discuss charge conjugation C and time reversal T via the space $\mathbb{C}^4$. It would of course be better to discuss them with the tangent objects. Neither can a definite charge be attached to R or L alone nor do the parameters a and $\tilde{a}$ alone have the meaning of "times" before they are identified according to equation (76). Nevertheless, we do find some kind of "charge transfer" from the two conjugate parts of the A individuals to the operation C in $\mathbb{R}^4_1$, and some kind of "time transfer" from the "times" τ in $U(2)$ via $\mathbb{C}^4$ to t of $\mathbb{R}^4_1$.

Consider the situation after a P transformation and try to restore the original situation using C and T. Charge conjugation in the $U(2)$ level corresponds to $\psi_1 \leftrightarrow \psi_2$ in Table II, which means that

$$c \leftrightarrow d \quad \text{in} \quad \mathbb{H} \qquad \text{and} \qquad \tilde{c} \leftrightarrow \tilde{d} \quad \text{in} \quad \tilde{\mathbb{H}} \tag{94}$$

in the $\mathbb{C}^4$ level, which corresponds to $x \leftrightarrow y$ (?) according to Table III. This operation does not interchange $\mathbb{H}$ and $\tilde{\mathbb{H}}$. "Time" τ reversal $(\tau \to -\tau)$ in the $U(2)$ level leads to $u_4 \to -u_4$ [cf. equation (15)], which means that

$$\tilde{\mathbb{H}} \to -\tilde{\mathbb{H}} \tag{95}$$

whereas $\mathbb{H}$ remains unchanged (!). In terms of equation (89), we have

$$(a, -ib, -ic, -id) \to (-a, +ib, +ic, +id),$$

an operation that does not interchange $\mathbb{H}$ and $\tilde{\mathbb{H}}$ either. And yet, in order to reach the situation before P, we have to interchange $\mathbb{H}$ and $\tilde{\mathbb{H}}$. This implies $a \to -a$, and this therefore corresponds to time reversal T in $\mathbb{R}_1^4$.

Remark 11.4. (Time.) The relation between the two τ's in the A individuals of $U(2)$ and the time t of $\mathbb{R}_1^4$ is hidden by the algebra $\mathfrak{C}_+$ [see equation (34)]. The process that "convolutes" the two τ's to the common time t (dynamical part of mass generation) is not known, although a first primitive approach to a simple "thermodynamic" variant was published some years ago (Donth, 1982).

As we have complex factors containing $i\omega\tau$ in the individuals, we arrive at Wigner's time reversal for $\tau \to -\tau$ (ψ and ψ^*), and at Born's bilinear probability construction ($\psi\psi^*$).

As mentioned above, the two a coordinates of $\mathbb{C}^4$ alone do not represent a time. But if one projects the time concept t from $\mathbb{R}_1^4$ to the two a's (see also Remark 11.3 relating to this point), then the two "apparent times" a, $\tilde{a}$ of Fig. 1 also go in opposite (Feynman) directions ($t = a = -\tilde{a}$).

The principal nonlocality of the individuals and of single vacuum elements enables one to construct a *physical* concept of a common time, which is also common for parts of the world that are temporarily "isolated" from one another.

Remark 11.5. (Boost B and rotation R.) The difference between the vector transformation (50) of the spinor space and the transformation of the tangent objects in $\mathbb{R}_1^4$ as represented by the matrix theorem can also be demonstrated by means of boost and rotation operators.

For the sake of simplicity, only the well-known infinitesimal transformation in the z direction is considered. Then, for real parameters α_1, α_2,

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1+\alpha_1+i\alpha_2 & 1 \\ 1 & 1-\alpha_1-i\alpha_2 \end{pmatrix} \quad \text{for} \quad \mathbb{C}^2 \tag{96}$$

$$A^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} = \begin{pmatrix} 1+\alpha_1-i\alpha_2 & 1 \\ 1 & 1-\alpha_1+i\alpha_2 \end{pmatrix} \quad \text{for} \quad \dot{\mathbb{C}}^2 \tag{97}$$

The generators I_1, I_2 are obtained by

$$\begin{aligned} A &= 1 + I_1\alpha_1 + I_2\alpha_2 + \cdots \\ A^* &= 1 + I_1\alpha_1 - I_2\alpha_2 + \cdots \end{aligned} \tag{98}$$

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B_z, \qquad I_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = R_z \tag{99}$$

I_3, I_4, I_5 and I_6 can be obtained analogously.

It is also well known that a classification of the Lorentz group representations (σ_z, σ'_z) is obtained by a nongeometrical (using imaginary coefficients) construction beyond the Lie algebra,

$$\sigma_z \sim I_1 + iI_2, \qquad \sigma'_z \sim (I_1 - iI_2) \tag{100}$$

with

$$\psi_L = (\tfrac{1}{2}, 0), \qquad \psi_R = (0, \tfrac{1}{2}) \tag{101}$$

where σ_z and σ'_z are linked to two mutually commutating $su(2)$ algebras (spin).

We must now ask: can σ_z and σ'_z be linked one-to-one with the two spaces $\mathbb{C}^2$ and $\dot{\mathbb{C}}^2$, respectively?

The answer is no for the transformation (50) of the spinor spaces, because the partition (98) is different from the partition

$$1 + \chi_1(I_1 + iI_2), \qquad 1 + \chi_2(I_1 - iI_2) \tag{102}$$

This is also clear from Remark 5.1: In general, the two-u construction distributes one individual onto both the $\mathbb{C}^2$ spaces.

But the answer is yes for the transformation of tangent objects mediated by the matrix theorem. With respect to the algebra $M(4, \mathbb{C})$, the χ's become complex parameters [cf. this with s in the equations (53) and (54) or with T in equations (67) and (68)]. Boost and rotation of the ψ_R objects are parametrized by $\operatorname{Re}\chi_1$ and $\operatorname{Im}\chi_1$, whereas $\operatorname{Re}\chi_2$ and $\operatorname{Im}\chi_2$ parametrize boost and rotation of the ψ_L objects.

The boost and rotation concept is therefore connected with the identity of the tangent objects, and not with the spinors as vectors in the $\mathbb{C}^2$ spaces. But the fact that boost and rotation can actually be divided into ψ_R and ψ_L objects is a remarkable quality of the general construction.

Remark 11.6 [Identity (iv).] The identity (i) problem of Remark 5.2 cannot be solved within the framework of our $\mathbb{R}^4_1$ construction. Essentially, so far we only have tangent space constructions for $\mathbb{R}^4_1$ and a necessary (but not sufficient) condition for identifying tangent objects (Section 7) that is also totally restricted to the uncertainty of the tangent constructions. We have not yet constructed any point, or, in other words, we do not have a "*Verpflanzung*" of the metric (Weyl, 1923, Chapters 7 and 8) or a space connection that would transform $\mathbb{R}^4_1$ into $\mathbb{M}_4$. Ultimately, the junction (76) only makes it possible to state that the conjugate particles have an equal amount of mass and charge. The main problem is to connect the rather independent spinors ψ_L and ψ_R, or more precisely their tangent objects, by connections, in order to define the particle identity during processes (movements with interactions), too. Phenomenologically, the "affine" connection links ψ_L and ψ_R by Dirac's mass form, and the gauge connection links ψ_R with ψ_R, and independently (!) ψ_L with ψ_L by the charge form(s) of simple field theories.

Remark 11.7. [Identity (v), "Majorana problem."] Originally, at the $U(2)$ level, the junction was defined between charge conjugate parts, plus and minus, but the spaces which we have constructed largely separate R and L.

12. OVERVIEW OF THE CONSTRUCTION

From the standpoint of a hidden $U(2)$ world as being the presumed reality behind the quantum mechanical formalism, the following space-time elements have been constructed from local tangent elements at the $U(2)$ individuals.

$\mathfrak{M}$ ordinary tangent space at a four-dimensional individual. There is no mention that its topology is $U(2)$.

$(q, \tilde{q})$, $(\mathbb{H}, \tilde{\mathbb{H}})$ biquaternions. Orthogonal two-u residue class construction over two conjugate $\mathfrak{M}$'s with respect to the wave metric of the individuals. Second Clifford algebra $\mathfrak{C}_+$.

$\mathbb{C}^4 = \mathbb{C}^2 \oplus \dot{\mathbb{C}}^2$ spinor space isomorphous to the biquaternions as a representative of the "between" of two conjugate individuals. The spinors ψ live here as nonspecified vectors (wave functions).

$SL(2, \mathbb{C})$ comes from a conformal comparison of $\mathbb{R}^2$ tangent spaces at two individuals. $\mathbb{R}^2$ is marked out by the fact that there are only two "physical" coordinates φ_1, φ_2 on $U(2)$ that carry potentials as a consequence of the electromagnetic wave transversality theorem.

$\{\sigma^k\} = \subset u(2)$ Hermitian matrix units which enable one to identify $U(2)$ individuals by their Lie algebra. Possibility of identifying tangent objects with a $U(2)$ structure, as studied from a common space.

$M(4, \mathbb{C})$ Dirac's Clifford algebra. Two aspects are considered here. (1) That is, too, a residue class construction with respect to the wave metric. (2) The second Clifford algebra can transform the tangential objects $x = \gamma^\mu$ or A_μ within the common tangent space $\mathbb{R}_1^4$ (TxT^{-1}).

$\mathbb{R}_1^4 = (\mathbb{H}, \mathbb{H}_c)$ a common tangent space (being, most likely, tangent to the common Minkowski point space $\mathbb{M}_4$, which remains to be constructed; this problem is discussed in Section 13). The eight dimensions of the biquaternions are reduced to four by the junction (76). A realizations [covariant vectors with values in the $u(2)$ Lie algebras] as well as spinors ψ (written as vectors $\bar{\psi}\gamma^\mu\psi$) can be represented within it.

Thus we have constructed $\mathbb{R}_1^4$ with living spaces for nontrivial $U(2)$ A and ψ individuals whose $U(2)$ structure is present, but completely hidden behind the general properties of $M(2, \mathbb{C})$ and $M(4, \mathbb{C})$. The *ad hoc* construction is "physical" and a very special one. It is more than a question of choosing between a set of similar spaces. A great deal of material (including algebraic material) is prepared for the construction of a quantum theory.

One can see from the numerous remarks that the logical relationships between the elements listed above are still rather complicated, probably for three reasons: (1) ψ, A, and ϕ are not "separated" in the $U(2)$ world, which is the cause for the interpenetration of the concepts based on them. (2) The $U(2)$ coordinates are well mixed by the two-u construction. (3) It is believed that a closure—the roof—of our construction cannot be reached before the problem of the space connections is solved.

The solution of three problems (among many others) seems to be most pressing: (1) Find the local symmetry group of the conjugate equations (1a), (1b); (2) find the differential geometric basis for the two-u construction; and (3) do similarly for the junction (76) and Fig. 1 (remember that the many vacuum elements are important for problems 2 and 3).

Remark 12.1. (Metric.) Our construction has two properties: The metric signature in $U(2)$ and $\mathbb{R}_1^4$ is the same, and the transverse potentials are linearly transferred from $U(2)$ via $\mathbb{C}^4$ to $\mathbb{R}_1^4$. This is prerequisite for the occurrence of linear wave equations in $U(2)$ and $\mathbb{M}_4$ (for A and ψ in the latter). The solution of the above three problems should answer to the question of degree up to which the metric and wave transfer are equivalent (i.e., is there a new kind of Huygens principle that permits a unique selection of the metric?).

I remark once more that the intermediate story $\mathbb{C}^4$ does not have such a metric. There is no independent electrodynamics in $\mathbb{C}^4$.

13. MAKING THE NEXT STEPS MORE PRECISE

Further work is planned within the framework of the $U(2)$ program. Using the algebraic constructions of the present paper, some of the next problems can be formulated more precisely than before.

13.1. Connections

Finite shifts are needed to enlarge the junction (76) into connections (see also Remark 11.6). Finite shifts are usually obtained from local (algebraic) elements by exponential maps. Two exponentials on $U(2)$ coordinates are already contained in the potentials of our individuals as wave factors,

$$e^{i\omega\tau} \quad \text{and} \quad e^{im\varphi_j}, \quad j=1,2 \tag{103}$$

It is a natural idea to use them (and some relations regarding the tori involved) for constructing affine and gauge connections, respectively. Not only the potentials, but also the phase surfaces of de Broglie's phase correspondence theorem (de Broglie, 1925) are forms in $\mathbb{R}_1^4$. No special individuals (quanta) for gravitation are contained in the $U(2)$ model. Therefore, any quantization of gravitation can only start from the assumption that the affine connection emerges from the $U(1)$ factor $\exp(i\omega\tau)$ of ψ and A individuals and of ϕ vacuum elements. The effect must go via $\mathbb{C}^4$ and we can ask whether the "fifth force" (ranged in the 100 m scale) can be grasped as the quantum effect of gravitation.

13.2. Points

At present we do not have a precise construction of a point space ($\mathbb{M}_4$) that could serve as a basis space for the tangent space $\mathbb{R}_1^4$ constructed. A point space would also be needed as a basis for classical gauge fiber bundles. In a previous paper (Donth and Lange, 1986) the idea was suggested that one can obtain culminating points using an interaction of ψ (or A) individuals with an infinite number of vacuum elements.

It is expected that our tangent space construction ($\mathbb{C}^4, \mathbb{R}_1^4$) and the possibility of identifying tangent objects therein are relatively independent of the culminating point construction. This independence seems to be a reasonable basis for quantum mechanical uncertainty.

Such a subsequent culminating point construction should, to some degree, refer to the renormalization equation (16) and to the compactness of $U(2)$. Because of the reference to renormalization, the meaning of ψ and A functions is expected to be modified in some aspects; for instance, the conception of being functions of $x \in \mathbb{M}_4$: $\psi(x)$, $A(x)$.

The concept of uncertainty was related to objects living in tangent spaces (tangent objects of Section 8). The concept of the limit $k \to \infty$, necessary for a culminating point, was related to mass and charge values, to the vacuum, and to the (apparatus of an) experiment. It is therefore our aim to make a construction with a far-reaching separation of wave function (as an object of the tangent space $\mathbb{C}^4$) and a point, such as in Bell's phenomenological construction (Bell, 1984). The advantages of such a separation are the possibility of making a natural construction of Born's probability with no violation of the superposition, and of reaching a conceptual solution to the problem of how the wave function is "reduced" during the experiment (Bell, 1984).

13.3. Spin Statistics Theorem

The main aspect here is to modify Weyl's ideas about the *Verpflanzung* of the metric, mentioned above, by requirements resulting from the new space construction (including $\mathbb{C}^4$). In this sense, the spin statistics theorem supplies us with some information about the similarity expected between the tangent space and the basic point space desired; more precisely, information about the connection between neighboring points of $\mathbb{M}_4$ on one hand and the properties of the tangent space $\mathbb{R}^3 \subset \mathbb{R}_1^4$ on the other hand. That is true because the following condition is sufficient as proof of the spin statistics theorem: A local region of the three-dimensional point space $\mathbb{E}^3$ containing two points as representatives of two identical particles must be equivalent to their common tangent space $\mathbb{R}^3$ in such a way that an exchange of the two points in $\mathbb{E}^3$ is equivalent to a π rotation of both their tangent objects. [The theorem thus follows from the well-known transformation behavior of spinors under a π rotation (Donth, 1970, 1977).]

As this idea is only related to $\mathbb{E}^3$ and $\mathbb{R}^3$, we have a nonrelativistic approach to the spin statistics theorem (!).

It is a general property of our space construction that it can conceptionally unite common space aspects (residue class construction, connections between many particles) and individual aspects (configurational space for tangent objects). Comparing Remarks 11.3–11.5, it seems that PCT, Spin and Statistics, and all that can conceptually be grasped by some simple relations (equivalent relations) between a region of the point space and our tangent space properties.

13.4. Geometry of Quantum Mechanics

Gauge fiber bundles are useful geometric constructions for semiclassical theories with minimal interaction, but they are probably less useful for quantum theories. In our model the geometry of quanta is assumed to be represented by ψ individuals (2) and A individuals (9) both being electrodynamic constructions on $U(2)$ manifolds. In our mass construction proposal, the points of basis are obtained from an "experiment" that uses the many vacuum elements. This construction does not destroy the principal sources of uncertainty, as mentioned above. This independence of points (experiment) and tangent objects (quanta) demands a more "spongy" construction. A complete alternative would be that the "fibers" are not pinned to points on a basis but to the tangential objects which have been rather independently constructed from a common basis.

13.5. Interactions

There would be no point in constructing a principle for the interactions that has no relation to the $U(2)$ objects. Consider the two-u formulas (30) and (31), and Remark 5.1 about the origin of the u's in it. Then it is perhaps useful, in the "pullback" to the $U(2)$ world, to assume that the interaction is a "mistaking" on the part of the ψ and A individuals of their "own" $U(2)$ space (or its S^3 part) for the $U(2)$ (or S^3) spaces of vacuum elements ϕ which are of nearly the same kind and which are offered in very great numbers. This high "space degeneracy" has been expressed by binomials such as in equation (13). (An analogous change by mistake can perhaps be constructed between the ϕ elements—ϕ self-interaction, realization of a chemical equilibrium between all possible kinds of vacuum elements generating the "true vacuum.")

13.6. The Way toward Charge and Mass (A Trial)

In our biharmonic parametrization (4) the group element $g \in U(2)$ can be represented by the matrix

$$g = e^{i\tau}\begin{pmatrix} \cos\vartheta_3\, e^{-i\varphi_2} & \sin\vartheta_3\, e^{i\varphi_1} \\ -\sin\vartheta_3\, e^{-i\varphi_1} & \cos\vartheta_3\, e^{i\varphi_2} \end{pmatrix} \tag{104}$$

The corresponding left and right invariant vector fields (L_i, R_i) are ($\varphi := \varphi_1 + \varphi_2$, $\psi := \varphi_1 - \varphi_2$)

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}:\quad L_1 = \cos\vartheta_3 \cos\varphi \frac{\partial}{\sin\vartheta_3\, \partial\varphi_1} - \sin\vartheta_3 \cos\varphi \frac{\partial}{\cos\vartheta_3\, \partial\varphi_2} + \sin\varphi \frac{\partial}{\partial\vartheta_3}$$

$$R_1 = \cos\vartheta_3 \cos\psi \frac{\partial}{\sin\vartheta_3\, \partial\varphi_1} + \sin\vartheta_3 \cos\psi \frac{\partial}{\cos\vartheta_3\, \partial\varphi_2} + \sin\psi \frac{\partial}{\partial\vartheta_3} \tag{105a}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}:\quad L_2 = -\cos\vartheta_3 \sin\varphi \frac{\partial}{\sin\vartheta_3\, \partial\varphi_1} + \sin\vartheta_3 \sin\varphi \frac{\partial}{\cos\vartheta_3\, \partial\varphi_2} + \cos\varphi \frac{\partial}{\partial\vartheta_3}$$

$$R_2 = -\cos\vartheta_3 \sin\psi \frac{\partial}{\sin\vartheta_3\, \partial\varphi_1} - \sin\vartheta_3 \sin\psi \frac{\partial}{\cos\vartheta_3\, \partial\varphi_2} + \cos\psi \frac{\partial}{\partial\vartheta_3} \tag{105b}$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}: \quad L_3 = -\frac{\partial}{\partial\varphi_1} - \frac{\partial}{\partial\varphi_2} \left(= -\sin\vartheta_3 \frac{\partial}{\sin\vartheta_3\,\partial\varphi_1} - \cos\vartheta_3 \frac{\partial}{\cos\vartheta_3\,\partial\varphi_2} \right) \tag{105c}$$

$$R_3 = +\frac{\partial}{\partial\varphi_1} - \frac{\partial}{\partial\varphi_2}$$

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}: \quad L_4 = R_4 = \frac{\partial}{\partial\tau} \tag{105d}$$

The seven vectors $\{L_4 = R_4, L_i, R_i\}$, $i = 1, 2, 3$, can serve as a basis for the seven Killing vectors of $U(2)$ with the standard metric.

The following difference seems to be important. The biharmonic coordinates (expressing the two-torus symmetry of the Heegaard splitting, and which were used in the two-u construction of the common tangent spaces) are true coordinates, that is,

$$\partial/\partial\varphi_1, \partial/\partial\varphi_2, \partial/\partial\vartheta_3, \partial/\partial\tau \quad \text{commute with one another} \tag{106}$$

whereas the vector fields (105a)-(105d) (expressing the S^3 and S^1 group symmetry) are not a coordinate basis on $U(2)$:

$$[L_i, L_j] = -2L_k, \qquad [R_i, R_j] = -2R_k \tag{107a}$$

$$[L_4, L_i] = 0, \qquad [R_4, R_i] = 0 \tag{107b}$$

($i = 1, j = 2, k = 3$ and cyclic permutations). The misfit of the corresponding "congruences" is given by these commutators, and one can construct curvatures from them.

Let us assume that the following principles must be used for the construction of the Lagrangian $\mathscr{L}$ (its gauge invariance expresses the S^1 and S^3 symmetry):

I. We must use the L_i (or R_i) as carriers of these symmetries.
II. The differences to coordinate systems must be compensated by additional terms.
III. The generating relations (30), $i \neq j$, for the Clifford algebra $\mathfrak{C}_+$ must come into a certain correspondence ($\sim$) to the commutators (107), for instance

$$u_1 u_2 + u_2 u_1 = 0 \sim [L_1, L_2] = -2L_3 \tag{108}$$

Then the electrical charge e_0 is to be attached to $[L_1, L_2]$ and the mass to $[L_1, L_3]$ and/or $[L_2, L_3]$.

Example 13.1. (Sketch of a possibility.) First consider charge. Let us assume that L_1, L_2 (or R_1, R_2) in $U(2)$ correspond to the covariant derivations ∇_j in $\mathbb{M}_4$. The angles φ_1 and φ_2 are the physical coordinates [carrying potentials on $U(2)$]. Taking the β isotropy mechanism discussed in Table III into account, we should expect to get all four ∇'s only from L_1 and L_2.

Then, considering the commutator applied to ψ, we have

$$[L_1, L_2] \to [\nabla_j, \nabla_k] \approx F_{jk} \tag{109}$$

We obtained the generator for the Maxwell field F as a gauge curvature. Now consider this commutator directly applied to the $U(2)$ individuals. Then we get their electrical charges m:

$$(1/2)[L_1, L_2] = -L_3 = \partial/\partial\varphi_1 + \partial/\partial\varphi_2 = \begin{Bmatrix} im & \text{for } \psi \\ i(m_1 + m_2) & \text{for } A \end{Bmatrix} \sim Q \tag{110}$$

Comparing equations (109) and (110), we see that the Maxwell fields are generated iff the charge is different from zero, $m \neq 0$. This means that our m-charged individuals really have *electrical* charges Q in the sense of Coulomb.

Analogously, we can speculate on how the commutators $[L_1, L_3] = 2L_2$ and $[L_2, L_3] = -2L_1$ lead to the masses. Contrary to L_3 for charge, L_1 and L_2 contain the hidden vectors $\partial/\partial\vartheta_3$ (the coordinate ϑ_3 does not carry potential, $A_3 = 0$). It is a "supersymmetric" variable because it "makes" neither ψ's nor A's, because ϑ_3 is "contained" in both [cf. equations (2) and (9)], and because ϑ_3 must "explain" why the vacuum elements ϕ [only depending on the ϑ_3 (and τ) strings] not only build a spinor sea, but also have many aspects of scalars. Because ϑ_3 is hidden, the way to the (Einstein?) curvature is also hidden and not so simple.

The charge value e_0 (or its square $\alpha = e_0^2/\hbar c$) is universal because ϑ_3 is not involved in L_3, whereas the mass values m_0 are not universal. They depend on the kind (m, k) of the individual, because ϑ_3 is involved in L_1, L_2, and in the $\tilde{y}_{mk}(x)$ functions, $x = \cos\vartheta_3$, which are different for different individuals. From equation (11) we see

$$\begin{aligned}
e&: \quad \tilde{y}_{11}(x) = x(1-x^2) \\
\nu_e&: \quad \tilde{y}_{01}(x) = (1-x^2) \\
\mu&: \quad \tilde{y}_{12}(x) = x(2-6x^2+4x^4) \\
\nu_\mu&: \quad \tilde{y}_{02}(x) = (1-4x^2+3x^4) \\
\tau&: \quad \tilde{y}_{13}(x) = x(3-18x^2+30x^4-15x^6) \\
\vartheta_\tau&: \quad \tilde{y}_{03}(x) = (1-9x^2+18x^4-10x^6) \\
&\quad \dots
\end{aligned} \tag{111}$$

$$\begin{aligned}
B&: \quad \tilde{y}_{10}(x) = x \\
E_0^{--}&: \quad \tilde{y}_{20}(x) = x^2, \qquad E_1^{--}: \quad \tilde{y}_{21}(x) = x^2(1-x^2), \dots
\end{aligned} \tag{112}$$

Two tendencies can be observed with increasing generation number k: (a) the maximal power increases with $m + 2k = \omega$, and (b) the number of zeros (oscillations, "statistical weights" of the binomials) increases with $k(k, \binom{k}{k/2}^2)$.

If we apply these tendencies to the mass values for the charged leptons, for instance, we see that (a′) each power of x gives a factor of order $\alpha^{-1/2}$ [a charge m is characterized, besides the exponential $\exp(im\varphi_j)$, by a universal *charge factor* x^m in $\tilde{y}_{mk}(x)$; assuming that this charge factor translates x to α and that a renormalization condition for it in the case $m=1$ can be represented by

$$g^\kappa x^\kappa \sim 1, \qquad \kappa \to \infty \tag{113}$$

[cf. equations (14) and (16)], then follows that $x \approx g^{-1}$; the statement (a′) is obtained from $g \approx \alpha^{1/2}$]; and (b′) the growing binomials damp the mass values of higher generations (since the vacuum $k \to \infty$ has the mass value zero).

The corresponding generation problem for the charged leptons is represented in Fig. 2. The breakdown of the (originally infinite) k series follows from the stability requirement that a particle of higher complexity (higher k) should have a higher mass value.

Let us go back to Fig. 1 or Table III. We observe that the $u_i u_j$ content of the right (R) and left (L) spinor space is rather different. According to the principle III above, nontrivial commutators (107a) can only be attached to R. We have the following result:

Mass values m_0 and charge e_0 can only be carried by particles with right-handed components.

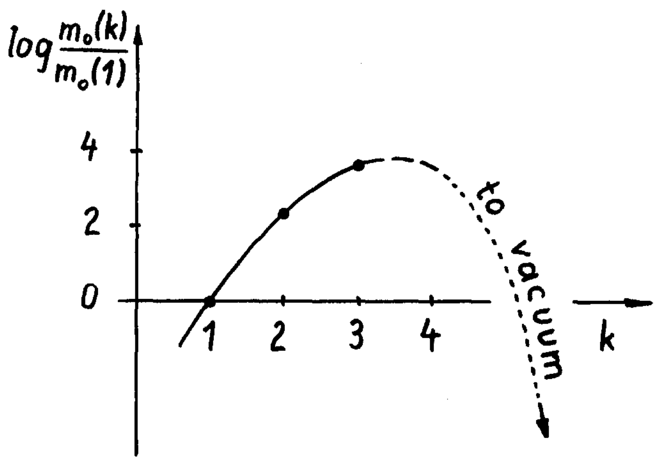

Fig. 2. A possible explanation for the existence of only a few lepton generations; m_0, mass values; k, generation number.

Only the left space L contains u_4 connected with the hidden τ time. It is therefore L that makes the dynamics. So R and L are quite unequal brothers that are united by the junction (76) to the common tangent space $\mathbb{R}_1^4$.

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REFERENCES

Baylis, W. E. (1980). *American Journal of Physics*, **48**, 918.

Baylis, W. E., and Jones, G. (1988). *Journal of Mathematical Physics*, **29**, 57.

Bell, J. (1984). In *Quantum, Space and Time—The Quest Continues*, A. O. Barut, A. van der Merwe, and J.-P. Vigier, eds., Cambridge University Press, Cambridge, p. 66.

Brauer, R., and Weyl, H. (1935). *American Journal of Mathematics*, **57**, 425.

Courant, R. (1962). *Partial Differential Equations*, Interscience, New York.

Cramer, J. G. (1986). *Review of Modern Physics*, **58**, 647.

De Broglie, L. (1925). *Annales de Physique*, **3**, 22.

Dirac, P. A. M. (1958). *The Principles of Quantum Mechanics*, 4th ed., Clarendon Press, Oxford.

Donth, E. (1970). *Physics Letters*, **32A**, 209.

Donth, E. (1977). *Wissenschaftliche Zeitschrift der Technischen Hochschule Leuna-Merseburg*, **19**, 602.

Donth, E. (1982). *Wissenschaftliche Zeitschrift der Technischen Hochschule Leuna-Merseburg*, **24**, 475.

Donth., E. (1984). *Experimentelle Technik der Physik*, **32**, 213.

Donth, E. (1986). *Wissenschaftliche Zeitschrift der Technischen Hochschule Leuna-Merseburg*, **28**, 280.

Donth, E., and Lange, O. (1986). *International Journal of Theoretical Physics*, **25**, 1235.

Dubrovin, B. A., Novikov, S. P., and Fomenko, A. T. (1986). *Sovremennaya Geometriya*, 2nd ed., Nauka, Moscow.

Einstein, A. (1909). *Physikalische Zeitschrift*, **10**, 185.

Fermi, E. (1954). *Notes on Quantum Mechanics*, University Press, Chicago.

Imaeda, K. (1976). *Nuovo Cimento*, **32B**, 138.

Penrose, R. (1968). *Structure of Space-Time*, Benjamin, New York.

Salingaros, N. (1981). *Journal of Mathematical Physics*, **22**, 226.

Schrödinger, E. (1940). *Proceedings of the Royal Irish Academy*, **46A**, 32.

Van der Waerden, B. L. (1967). *Algebra II*, 5th ed., Springer, Berlin.

Van der Waerden, B. L. (1971). *Algebra I*, 8th ed., Springer, Berlin.

Van der Waerden, B. L. (1974). *Group Theory and Quantum Mechanics*, Springer, Berlin.

Von Weizsäcker, C. F. (1986). *Aufbau der Physik*, 2nd ed., Hanser, München.

Weyl, H. (1923). *Mathematische Analyse des Raumproblems*, Springer, Berlin.